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The Lipkin model. Beyond mean field with generalized coherent states

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Abstract

The Schwinger representation and the Marumori–Yamamura–Tokunaga boson expansion are used to describe the Lipkin model in terms of generalized coherent states. The ground-state energy is obtained within several variant types of coherent states. It has been found that two of the kinds considered describe particularly well the transition from weak to strong coupling, providing a remarkable improvement of the mean-field description of the transition zone. The time evolution predicted by generalized coherent states has been investigated numerically by comparing its dynamics with the exact one.

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1. Introduction

The Lipkin model was originally proposed by Lipkin, Meshkov and Glick [1] and has been widely used to test several kinds of many-body theories of strongly interacting fermion systems, such as, for instance, the time-dependent Hartree–Fock method [2], finite temperature dynamics [3, 4], collective dynamics of many-fermion systems [5, 6], phase transitions, spin tunelling [7], etc. It is defined as a system having two levels in a fixed shell-model potential with the same j-value, one just below and the other just above the Fermi level. In the reference state, relative to which excitations are measured, the lower level is filled with $2j + 1$ fermions and the upper level is empty. Each magnetic quantum number m, $-j \leqslant m \leqslant j$, is occupied either in the lower level or in the upper level, so that fermions jump, to and fro, between the two levels, maintaining the m-value. The Lipkin model is an $su(2)$ algebraic model, which means that its Hamiltonian is expressed in terms of the generators of the $su(2)$ algebra realized as appropriate bilinear forms of fermion annihilation and creation operators, a structure which renders the model amenable to an exact treatment and makes it suitable to test theories and approximation methods. The model and some of its applications are reviewed in [8, 9].

A mapping, defined in the framework of the Marumori–Yamamura–Tokunaga (MYT) boson expansion [10], from a 'physical' boson space onto a 'model' boson space, was proposed in [11] to describe multiphonon processes. The same kind of mapping was used in [12] to study the Lipkin model [1] expressed in the Schwinger realization of the $su(2)$ algebra. In particular, a multiboson coherent state suitable to be used as a trial function in the time-dependent variational method was introduced. The new coherent state has the form of the extended coherent states discussed in [13], and in fact can be interpreted in terms of the deformed boson scheme [11, 14–17].

In the present paper, we test the generalized coherent states introduced in [12, 18] by studying the ground-state energy and the time evolution of a system described by the Lipkin model. In the mean-field approximation this model contains a second-order phase transition from a weak coupling regime to a strong coupling regime. We will also show that the new generalized coherent states defined in Hilbert subspaces with fixed particle numbers, the eigenspaces of the particle number operator, reproduce well the behaviour of the transition region from the weak to the strong coupling regimes.

The ground-state energy of the Lipkin model was recently obtained by Tsue, Azuma, Kuriyama and Yamamura, only by use of the quasi-spin coherent state [19]. The dynamics of the q-deformed Lipkin model has been discussed in [20].

The paper is organized as follows: in section 2 the Hamiltonian in the new boson space is derived using the MYT boson expansion method. In section 3, we introduce new generalized coherent states defined in restricted Hilbert spaces with finite particle numbers. The ground-state energy and the time evolution are calculated. Finally, we present some numerical results and draw some conclusions.

2. Lipkin model: Schwinger realization

The Lipkin model is a schematic model defined by the Hamiltonian

$$\hat{H} = 2cS_0 + g\left(S_+^2 + S_-^2\right). \tag{1}$$

The operators $S_0, S_\pm$ are generators of the $su(2)$ algebra: $[S_0, S_\pm] = \pm S_\pm$, $[S_+, S_-] = 2S_0$, and c, g are positive parameters. The Hilbert space of the model is a proper invariant subspace associated with a specific eigenvalue of the Casimir operator specified by the multiplicity $(2j+1)$ of the single particle orbitals.

In the Schwinger realization, the generators of the $su(2)$ algebra are expressed in terms of two kinds of bosons $(a^\dagger, a)$ and $(b^\dagger, b)$ by

$$S_+ = a^\dagger b \qquad S_- = b^\dagger a \qquad S_0 = (a^\dagger a - b^\dagger b)/2 \tag{2}$$

and the Casimir operator reads

$$\hat{P} = (a^\dagger a + b^\dagger b)/2. \tag{3}$$

Substituting (2) in the Hamiltonian (1) we get (with $c = 1$)

$$H = a^\dagger a - b^\dagger b + g(a^{\dagger 2} b^2 + b^{\dagger 2} a^2) \qquad [a, a^\dagger] = [b, b^\dagger] = 1. \tag{4}$$

2.1. Semiclassical approach

The time-dependent Schrödinger equation

$$\mathrm{i}|\dot{\Psi}\rangle = H|\Psi\rangle \tag{5}$$

where $|\dot{\Psi}\rangle = \partial_t|\Psi\rangle$, may de derived from the action principle

$$\delta \int \left(\frac{\mathrm{i}}{2}(\langle\Psi|\dot{\Psi}\rangle - \langle\dot{\Psi}|\Psi\rangle) - \langle\Psi|H|\Psi\rangle \right) = 0. \tag{6}$$

This property is useful to obtain the 'optimal' time evolution of approximate wave packets $|C\rangle$ [21, 22], and is the basis of the time-dependent Hartree–Fock method, $|C\rangle$ being, in this case, a Slater determinant [23].

The semiclassical version of Hamiltonian (4) is

$$\begin{aligned} H_{cl} &= \langle C|\hat{H}|C\rangle \\ &= \alpha^*\alpha - \beta^*\beta + g(\alpha^{*2}\beta^2 + \beta^{*2}\alpha^2) \qquad \{\alpha^*, \alpha\} = \{\beta^*, \beta\} = -\mathrm{i} \end{aligned}$$

where α, β are complex numbers. The semiclassical Hamiltonian H_{cl} is the expectation value of (4) for a Glauber coherent state, $|C\rangle = \mathcal{N} \exp(\alpha a^\dagger + \beta b^\dagger)|0\rangle$, with $\mathcal{N} = \exp\left(-\frac{1}{2}(\alpha^*\alpha + \beta^*\beta)\right)$. The Glauber coherent state $|C\rangle$ will have, unavoidably, components outside the physical subspace, even for suitable values of the semiclassical Casimir operator $P_{cl} = \langle C|\hat{P}|C\rangle$. The semiclassical Lagrangian is

$$\begin{aligned} -\mathcal{L} &= -\frac{\mathrm{i}}{2}(\langle C|\dot{C}\rangle - \langle\dot{C}|C\rangle) + \langle C|\hat{H}|C\rangle \\ &= -\frac{\mathrm{i}}{2}(\alpha^*\dot{\alpha} - \dot{\alpha}^*\alpha + \beta^*\dot{\beta} - \dot{\beta}^*\beta) + \alpha^*\alpha - \beta^*\beta + g(\alpha^{*2}\beta^2 + \beta^{*2}\alpha^2). \end{aligned} \tag{7}$$

It is clear that $2P_{cl} = \alpha^*\alpha + \beta^*\beta$ is a constant of motion. We may replace H_{cl} by

$$H' = \alpha^*\alpha - \beta^*\beta + g(\alpha^{*2}\beta^2 + \beta^{*2}\alpha^2) - \lambda(\alpha^*\alpha + \beta^*\beta)$$

and look for the fixed points,

$$\begin{aligned} \frac{\partial H'}{\partial\alpha} &= (1-\lambda)\alpha^* + 2g\beta^{*2}\alpha = 0 \\ \frac{\partial H'}{\partial\beta} &= -(1+\lambda)\beta^* + 2g\alpha^{*2}\beta = 0. \end{aligned}$$

We find

$$\frac{\alpha\beta^{*2}}{\alpha^*} = \frac{\lambda - 1}{2g} \qquad \frac{\beta\alpha^{*2}}{\beta^*} = \frac{\lambda + 1}{2g}.$$

Either $\arg(\alpha) = \arg(\beta)$ so that $\alpha\beta^*/(\alpha^*\beta) = 1$, or $\arg(\alpha) = \arg(\beta) + \pi$, and $\alpha\beta^*/(\alpha^*\beta) = -1$. In the first case we have

$$\alpha^*\alpha + \beta^*\beta = 2P = \frac{\lambda}{g}$$

while in the second case we have,

$$\alpha^*\alpha + \beta^*\beta = 2P = -\lambda/g.$$

We consider the first solution which corresponds to the ground state. Under the transformation

$$\alpha \to \sqrt{P + \frac{1}{2g}} + \alpha \qquad \beta \to \sqrt{P - \frac{1}{2g}} + \beta$$

the Hamiltonian undergoes the replacement

$$\begin{aligned} H' \to H' = \frac{1}{2g} &- 2P^2g + (1 - 2Pg)\alpha^*\alpha - (1 + 2Pg)\beta^*\beta \\ &+ \frac{1}{2}\Big[(2Pg - 1)(\alpha^{*2} + \alpha^2) + (2Pg + 1)(\beta^{*2} + \beta^2) \\ &+ 4\sqrt{4P^2g^2 - 1}(\alpha^*\beta + \beta^*\alpha)\Big] + \cdots. \end{aligned}$$

The ground-state energy is $E_0 = -\frac{1}{2g} - 2P^2g$. The upper state energy is $-E_0 = \frac{1}{2g} + 2P^2g$ and corresponds to $2P = -\lambda/g$.

2.2. MYT boson expansion

Following the MYT boson mapping discussed in [12] we introduce new boson operators A, B such that

$$a^\dagger a \to 2A^\dagger A \qquad b^\dagger b \to 2B^\dagger B$$
$$a^{\dagger 2} \to 2A^\dagger\sqrt{A^\dagger A + \tfrac{1}{2}} \qquad b^{\dagger 2} \to 2B^\dagger\sqrt{B^\dagger B + \tfrac{1}{2}}. \tag{8}$$

With this correspondence the image of the Hamiltonian (4) becomes

$$H = 2(A^\dagger A - B^\dagger B) + G\left(A^\dagger\sqrt{A^\dagger A + \tfrac{1}{2}}\sqrt{B^\dagger B + \tfrac{1}{2}}B + B^\dagger\sqrt{B^\dagger B + \tfrac{1}{2}}\sqrt{A^\dagger A + \tfrac{1}{2}}A\right) \tag{9}$$

with $G = 4g$.

2.2.1. Coherent state Ia. Consider the following type of coherent state

$$|CSIa\rangle = |c\rangle = \mathcal{N}\exp\left(WA^\dagger\sqrt{A^\dagger A + \tfrac{1}{2}}\right)\exp\left(VB^\dagger\sqrt{B^\dagger B + \tfrac{1}{2}}\right)|0\rangle \tag{10}$$

$$\mathcal{N}^2 = \sqrt{1 - W^*W}\sqrt{1 - V^*V} \qquad A|0\rangle = B|0\rangle = 0.$$

We observe that $|CSIa\rangle$ is a common eigenvector of the generalized annihilation operators $\left(\sqrt{A^\dagger A + \frac{1}{2}}\right)^{-1}A$ and $\left(\sqrt{B^\dagger B + \frac{1}{2}}\right)^{-1}B$,

$$\frac{1}{\sqrt{A^\dagger A + \frac{1}{2}}}A|CSIa\rangle = W|CSIa\rangle \qquad \frac{1}{\sqrt{B^\dagger B + \frac{1}{2}}}B|CSIa\rangle = V|CSIa\rangle. \tag{11}$$

Similar to the Glauber coherent state, the coherent state $|CSIa\rangle$ will also have components outside the physical subspace, even for appropriate values of $P_{cl} = \langle C|\hat{P}|C\rangle$.

With the choice [10], the Lagrangian becomes

$$\begin{aligned}
-\mathcal{L} &= -\frac{\mathrm{i}}{2}[\langle c|\dot{c}\rangle - \langle\dot{c}|c\rangle] + \langle c|H|c\rangle \\
&= -\frac{\mathrm{i}}{2}\left(\frac{W^*\dot{W} - \dot{W}^*W}{2(1 - W^*W)} + \frac{V^*\dot{V} - \dot{V}^*V}{2(1 - V^*V)}\right) \\
&\quad + 2\left(\frac{W^*W}{2(1 - W^*W)} - \frac{V^*V}{2(1 - V^*V)}\right) + G\frac{W^*V + V^*W}{4(1 - W^*W)(1 - V^*V)}.
\end{aligned}$$

As before, we introduce new pairs of variables (R_i, θ_i), $i = 1, 2$, such that $W = R_1\exp(\mathrm{i}\theta_1)$ and $V = R_2\exp(\mathrm{i}\theta_2)$. In terms of these variables we have

$$\begin{aligned}
-\mathcal{L} &= -\frac{R_1}{2(1 - R_1)}\dot{\theta}_1 - \frac{R_2}{2(1 - R_2)}\dot{\theta}_1 + 2\left(\frac{R_1}{2(1 - R_1)} - \frac{R_2}{2(1 - R_2)}\right) \\
&\quad + 2G\frac{\sqrt{R_1R_2}}{4(1 - R_1)(1 - R_2)}\cos(\theta_1 - \theta_2).
\end{aligned}$$

With $R_i = 2P_i/(1 + 2P_i)$, $i = 1, 2$, where

$$P_1 = \langle CSIa|A^+A|CSIa\rangle \qquad P_2 = \langle CSIa|B^+B|CSIa\rangle$$

we obtain

$$\begin{aligned}
-\mathcal{L} &= -P_1\dot{\theta}_1 - P_2\dot{\theta}_2 + 2(P_1 - P_2) + 2GP_1P_2\sqrt{\left(1 + \frac{1}{2P_1}\right)\left(1 + \frac{1}{2P_2}\right)}\cos(\theta_1 - \theta_2) \\
&\approx -P_1\dot{\theta}_1 - P_2\dot{\theta}_2 + 2(P_1 - P_2) \\
&\quad + 2G\left(P_1P_2 + \frac{1}{4}(P_1 + P_2) + \frac{1}{16} - \frac{1}{32}\left(\frac{P_1}{P_2} + \frac{P_2}{P_1}\right)\right)\cos(\theta_1 - \theta_2)
\end{aligned}$$

the last expression being valid for large P_i. The semiclassical Casimir operator $P_{cl} = P_1 + P_2$ is still a constant of motion.

2.2.2. Coherent state IIa. Consider now the following type of coherent state which exhibits a better convergence for large boson numbers

$$|CSIIa\rangle = |c\rangle = \mathcal{N}\exp\left(WA^\dagger\frac{1}{\sqrt{A^\dagger A+\frac{1}{2}}}\right)\exp\left(VB^\dagger\frac{1}{\sqrt{B^\dagger B+\frac{1}{2}}}\right)|0\rangle$$
$$\mathcal{N}^2 = \frac{1}{\cosh(2\sqrt{W^*W})\cosh(2\sqrt{V^*V})}. \qquad (12)$$

We observe that $|CSIIa\rangle$ is a common eigenvector of generalized annihilation operators $\sqrt{A^\dagger A+\frac{1}{2}}A$ and $\sqrt{B^\dagger B+\frac{1}{2}}B$,

$$\sqrt{A^\dagger A+\tfrac{1}{2}}A|CSIIa\rangle = W|CSIIa\rangle \qquad \sqrt{B^\dagger B+\tfrac{1}{2}}B|CSIIa\rangle = V|CSIIa\rangle. \qquad (13)$$

Again, similar to the Glauber coherent state, the coherent state $|CSIIa\rangle$ will also have components outside the physical subspace, even for appropriate values of the Casimir operator $P_{cl} = \langle C|\hat{P}|C\rangle$. The semiclassical Lagrangian becomes

$$\begin{aligned}-\mathcal{L} &= -\frac{\mathrm{i}}{2}(\langle c|\dot{c}\rangle - \langle\dot{c}|c\rangle) + \langle c|H|c\rangle \\ &= -\frac{\mathrm{i}}{2}\sqrt{W^*W}\tanh(2\sqrt{W^*W})\left(\frac{\dot{W}}{W}-\frac{\dot{W}^*}{W^*}\right) - \frac{\mathrm{i}}{2}\sqrt{V^*V}\tanh(2\sqrt{V^*V})\left(\frac{\dot{V}}{V}-\frac{\dot{V}^*}{V^*}\right) \\ &\quad + 2(\sqrt{W^*W}\tanh(2\sqrt{W^*W}) - \sqrt{V^*V}\tanh(2\sqrt{V^*V})) + G(W^*V+V^*W).\end{aligned}$$

In terms of the variables (R_i, θ_i), $i = 1, 2$, such that $W = R_1\exp(\mathrm{i}\theta_1)$ and $V = R_2\exp(\mathrm{i}\theta_2)$, we have

$$\begin{aligned}-\mathcal{L} &= -R_1\tanh 2R_1\dot{\theta}_1 - R_2\tanh 2R_2\dot{\theta}_2 \\ &\quad + 2(R_1\tanh 2R_1 - R_2\tanh 2R_2) + 2GR_1R_2\cos(\theta_1-\theta_2) \\ &\approx -P_1\dot{\theta}_1 - P_2\dot{\theta}_2 + 2(P_1-P_2) + 2GP_1P_2\cos(\theta_1-\theta_2)\end{aligned}$$

with $R_i \approx P_i(1+2\,\mathrm{e}^{-4P_i})$, valid for large R_i. The quantity $P = P_1+P_2$ is a constant of motion. The quantum fluctuations are reflected in the factors $\tanh 2R_1$, $\tanh 2R_2$. For large values of R_1, R_2 these factors are very close to unity ($\tanh(2) = 0.9640$, $\tanh(4) = 0.999\,329$).

Expressions for the wave packets $|CSIa\rangle$, $|CSIIa\rangle$ look different but they behave very similarly when the averages $P_1 = \langle A^\dagger A\rangle$, $P_2 = \langle B^\dagger B\rangle$ are large.

3. Conserving approximations

Mean-field dynamics, associated with the time evolution of a coherent state wave packet, deviates from the exact quantal dynamics in two respects: in the mean-field approach, decoherence effects are suppressed and quantum fluctuations associated with the conservation of constants of motion are also neglected. In our example, $\hat{P} = \frac{1}{2}(a^\dagger a + b^\dagger b) = A^\dagger A + B^\dagger B$ is a constant of motion which is only conserved on average since, as has been observed, the coherent states considered necessarily contain components outside the physical Hilbert space. Quantum fluctuations associated with the conservation of $\hat{P}$ may be taken into account if modified coherent states which belong to the physical subspace and, therefore, are themselves eigenstates of this operator are used to describe the dynamics. In the following we will consider two kinds of trial wavefunctions defined in a Hilbert subspace with a finite number of particles $2P$, namely, the eigenspace of $\hat{P}$ spanned by the states

$$|n_\alpha, n_\beta\rangle \qquad n_\alpha = 0, 1, \ldots, 2P \qquad n_\beta = 0, 1, \ldots, 2P \qquad n_\alpha + n_\beta = 2P$$

or

$$|n_A, n_B\rangle \qquad n_A = 0, 1, \ldots, P \qquad n_B = 0, 1, \ldots, P \qquad n_A + n_B = P.$$

3.1. The $su(2)$ coherent state

The $su(2)$ coherent state [24], belonging to the Hilbert space introduced above, is

$$|CS\rangle = |c\rangle = \mathcal{N}\exp(Va^\dagger b)|0, 2P\rangle \tag{14}$$

with V a complex number and

$$\mathcal{N} = \frac{1}{(1+V^*V)^P} \qquad a^\dagger a|0, 2P\rangle = 0 \qquad b^\dagger b|0, 2P\rangle = 2P|0, 2P\rangle.$$

According to Thouless' theorem [8], $|CS\rangle$ is a Slater determinant. For

$$V = \tan(\theta)\exp(\mathrm{i}\phi),$$

we get from (1)

$$\langle c|H|c\rangle = -2P\cos(2\theta) + gP(2P-1)\sin^2(2\theta)\cos(2\phi)$$
$$\frac{\mathrm{i}}{2}(\langle c|\dot{c}\rangle - \langle\dot{c}|c\rangle) = 2P\sin^2\theta\dot{\phi}.$$

The Lagrangian becomes

$$L = \frac{\mathrm{i}}{2}(\langle c|\dot{c}\rangle - \langle\dot{c}|c\rangle) - \langle c|H|c\rangle = \nu\dot{\phi} - \mathcal{H}(\nu, \phi) \tag{15}$$

where ν is the canonical conjugated variable of ϕ,

$$\nu = \langle c|a^\dagger a|c\rangle = 2P\sin^2\theta.$$

and

$$\mathcal{H}(\nu, \phi) = \langle c|H|c\rangle = 2(\nu - P) + 2g(2P-1)\nu\left(1 - \frac{\nu}{2P}\right)\cos(2\phi).$$

The classical dynamics of the system is described by the set of equations

$$\dot{\phi} = \frac{\partial\mathcal{H}}{\partial\nu} \qquad \dot{\nu} = -\frac{\partial\mathcal{H}}{\partial\phi}. \tag{16}$$

For $\varepsilon = \mathcal{H}$, we obtain

$$\dot{\nu}^2 = 4\left[\left(2g(2P-1)\nu\left(1 - \frac{\nu}{2P}\right)\right)^2 - (\varepsilon - 2(\nu - P))^2\right]. \tag{17}$$

The ground-state energy is

$$E_0 = \begin{cases} -2P & \chi' \leqslant 1 \\ -P(\chi' + 1/\chi') & \chi' > 1 \end{cases} \tag{18}$$

where $\chi' = (2P-1)g$. The coherent state (14) gives rise to a second-order phase transition from weak coupling regime (which corresponds to a harmonic energy spectrum) $\chi' < 1$ to a strong coupling regime $\chi' > 1$, corresponding to a rotation-like energy spectrum.

3.2. The $su(2)$ deformed coherent state I

Let $\mathcal{A} = \sqrt{A^\dagger A + \frac{1}{2}}A$ and $\mathcal{B} = \sqrt{B^\dagger B + \frac{1}{2}}B$, where the bosons A and B have been introduced in section 2. We define the coherent state

$$|CSI\rangle = |c\rangle = \mathcal{N}\exp(V\mathcal{A}^\dagger\mathcal{B})|0, P\rangle \tag{19}$$

where $A^\dagger A|0, P\rangle = 0$, $B^\dagger B|0, P\rangle = P|0, P\rangle$, V is a complex variable and

$$\mathcal{N}^{-2} = \sum_{n=0}^{P} C(n, |V|) \qquad C(n, |V|) = |V|^{2n}\frac{P!}{n!(P-n)!}\frac{\left[n - \frac{1}{2}\right]!\left[P - \frac{1}{2}\right]!}{\left[P - n - \frac{1}{2}\right]!}$$

with

$$\left[x-\tfrac{1}{2}\right]! = \left(x-\tfrac{1}{2}\right)\left(x-\tfrac{3}{2}\right)\left(x-\tfrac{5}{2}\right)\cdots\tfrac{1}{2}. \tag{20}$$

We get

$$\langle c|A^\dagger A|c\rangle = \mathcal{N}^2 \sum_{n=0}^{P} nC(n,|V|) = \nu \qquad \langle c|B^\dagger B|c\rangle = P-\nu.$$

The expectation value of the Hamiltonian (9) for this wave packet is

$$\langle c|H|c\rangle = \mathcal{H}(\nu,\phi) = 2(2\nu-P) + 8g\frac{\nu}{|V|}\cos\phi$$

where $\phi = \arg(V)$. From the Lagrangian (15) and the equations of motion (16), we obtain, for $\varepsilon = \mathcal{H}$,

$$\dot{\nu}^2 = \left[\left(8g\frac{\nu}{|V|}\right)^2 - (\varepsilon - 2(2\nu-P))^2\right]. \tag{21}$$

The ground-state energy is given by

$$E_0 = \mathcal{H}(\nu_0,\pi) \qquad \left(\frac{\partial\mathcal{H}}{\partial\nu}\right)_{\nu=\nu_0} = 0. \tag{22}$$

3.3. The su(2) deformed coherent state II

In the present case, we define the operators $\mathcal{A}'$ and $\mathcal{B}'$

$$\mathcal{A}' = \frac{1}{\sqrt{A^\dagger A + \frac{1}{2}}}A \qquad \mathcal{B}' = \frac{1}{\sqrt{B^\dagger B + \frac{1}{2}}}B.$$

We introduce the coherent state

$$|CSII\rangle = |c\rangle = \mathcal{N}' \exp(V\mathcal{A}'^\dagger\mathcal{B}')|0,P\rangle \tag{23}$$

where $A^\dagger A|0,P\rangle = 0$, $B^\dagger B|0,P\rangle = P$, V is a complex variable and

$$\mathcal{N}'^{-2} = \sum_{n=0}^{P} C'(n,|V|) \qquad C'(n,|V|) = |V|^{2n}\frac{P!}{n!(P-n)!}\frac{\left[P-n-\frac{1}{2}\right]!}{\left[n-\frac{1}{2}\right]!\left[P-\frac{1}{2}\right]!}$$

with $\left[x-\frac{1}{2}\right]!$ defined in (20). We get

$$\langle c|A^\dagger A|c\rangle = \mathcal{N}'^{-2}\sum_{n=0}^{P} nC'(n,|V|) = \nu \qquad \langle c|B^\dagger B|c\rangle = P-\nu. \tag{24}$$

The expectation value of the Hamiltonian is

$$\langle c|H|c\rangle = \mathcal{H}(\nu,\phi) = 2(2\nu-P) + \frac{8g}{|V|}\mathcal{F}(|V|)\cos\phi \tag{25}$$

where $\phi = \arg(V)$, and

$$\mathcal{F}(|V|) = \left\langle A^\dagger A\left(A^\dagger A - \tfrac{1}{2}\right)\left(P - A^\dagger A + \tfrac{1}{2}\right)\right\rangle$$

with

$$\langle f(A^\dagger A)\rangle = \mathcal{N}'^{-2}\sum_{n=0}^{P} f(n)C'(n,|V|).$$

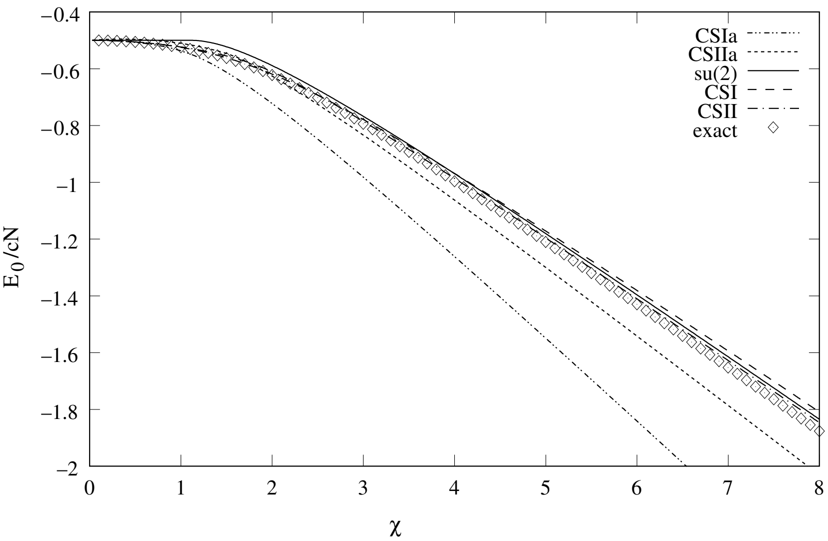

Figure 1. Ground-state energy as a function of the coupling constant for $N = 10$.

The classical dynamics of the system is described by the Lagrangian (15) and by the set of equations (16) in combination with (24) and (17). For $\varepsilon = \mathcal{H}$,

$$\dot{\nu}^2 = \left[\left(\frac{8g}{|V|} \mathcal{F}(|V|) \right)^2 - (\varepsilon - 2(2\nu - P))^2 \right]. \qquad (26)$$

The ground-state energy is given by equations (25) with $\mathcal{H}$ defined in (22).

4. Results and conclusions

In order to compare the performance of the generalized coherent states in a variational sense we have computed the ground-state energy of the system using those states as trial functions. In figure 1 we compare, for different values of the coupling constant $\chi = 2gP = gN$, with $N = 2P = 10$, the ground-state energy obtained with the five variants of coherent states (10), (12), (14), (19) and (23). The first conclusion is that the coherent states CSIa and CSIIa, respectively (10) and (12), which violate the particle number conservation, give results which are not so good, but which, nevertheless, are qualitatively acceptable. The particle number is only conserved on average by these states. It is clearly seen from figure 1 that they may even lead, for the energy, to values lying below the exact ground-state energy. This is not surprising, since the coherent states contain components outside the physical subspace and some of the eigenvalues of the Hamiltonian H extended to the enlarged space lying below the lowest eigenvalue associated with the physical subspace. It should be said, however, that CSIIa is quite reasonable up to and around the transition region, so that deformation of the coherent state apparently compensates for the lack of projection into states with good particle number. In spite of that we will only discuss in more detail the performance of the conserving coherent states, namely CS, CSI and CSII, respectively (14), (19) and (23), which belong to eigenspaces of the particle number or Casimir operator. In figure 2 we compare, for different values of the coupling constant χ, and for different particle numbers, the exact ground-state energy with

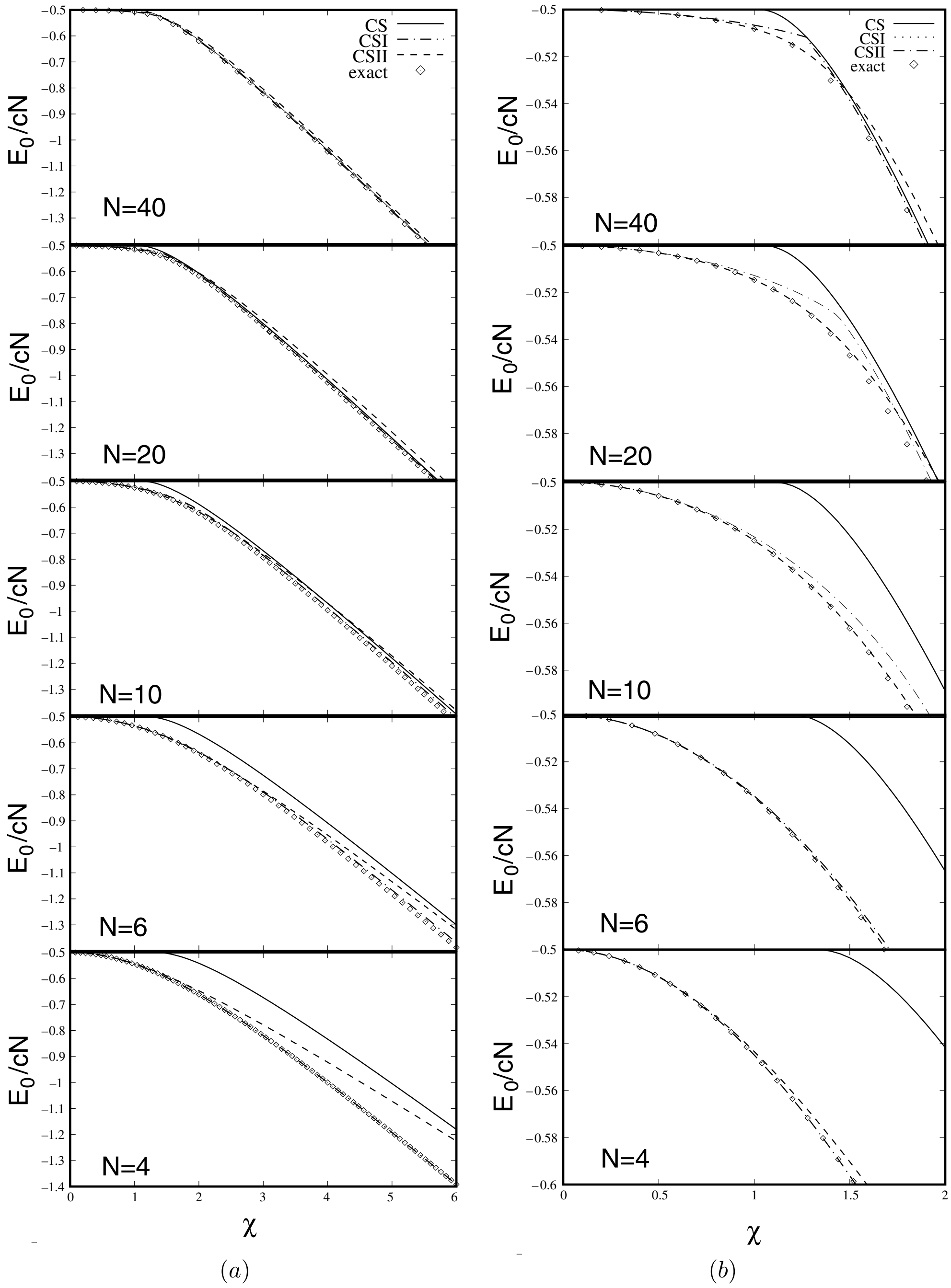

Figure 2. Ground-state energy as a function of the coupling constant for different numbers of particles.

the approximate values computed with the states CS, CSI and CSII. Several conclusions may be drawn.

- As expected, the larger the number of particles the better the approximate result.

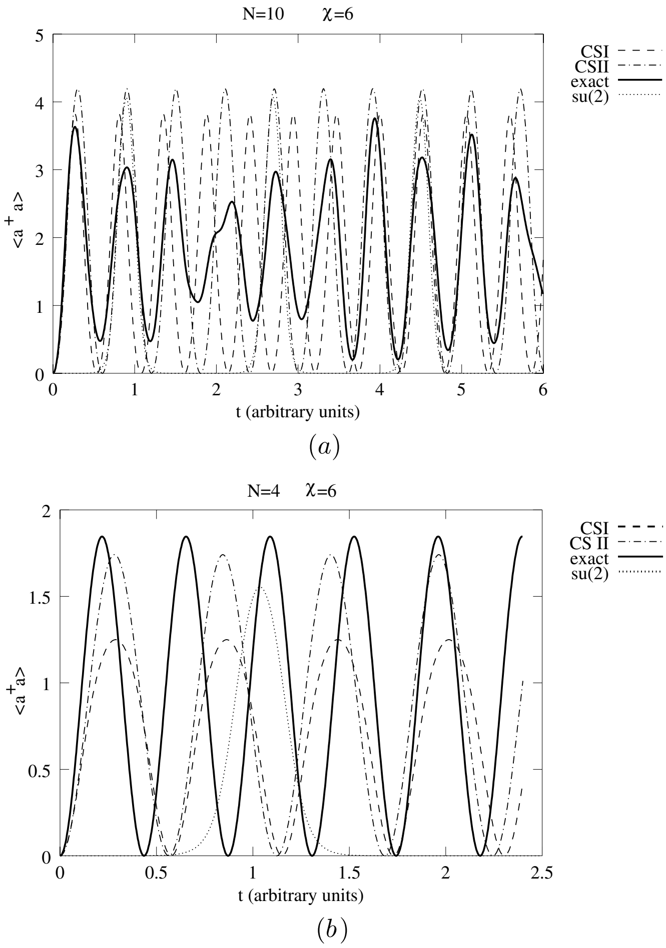

Figure 3. Time evolution with different coherent states for $\chi = 6$ and (*a*) $N = 10$; (*b*) $N = 4$.

- The generalized coherent states CSI and CSII do much better than the standard $su(2)$ coherent state CS for $\chi \sim 1$: in fact the second-order phase transition between the weak and the strong coupling regime is no longer present, the smoothness of the transition being reproduced.
- Although for values of $\chi \leqslant 2$ the coherent state CSI does better than CSII, for $\chi > 2$ the contrary is true. The smaller the number of particles the earlier CSI becomes worse.
- For large values of χ and large particle numbers the generalized coherent state CSI becomes worse than the standard $su(2)$ coherent state CS.
- The coherent state CSII is always better than the coherent state CS and is particularly good for small particle numbers. It is remarkable that CSII improves significantly the independent particle approximation or Slater determinant described by CS.

Next we analyse the time evolution of the system starting from an initial state containing no bosons of type a, i.e. at $t = 0$, the system is described by the ket $|0, 2P\rangle$. We show, in figures 3 and 4, the time evolution of the average number of bosons a, obtained by solving equations (17), (21), (26), for systems with four and ten particles and for $\chi = 6$ and $\chi = 2$.

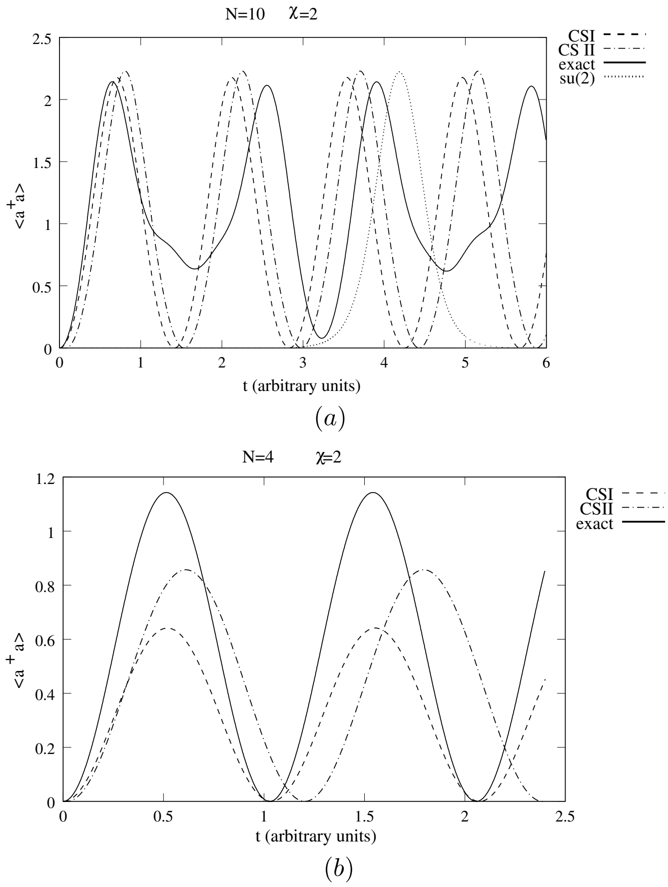

Figure 4. Time evolution with different coherent states for $\chi = 2$ and (*a*) $N = 10$; (*b*) $N = 4$.

For $N = 10$, the exact time evolution shows some features suggesting revival and decoherence effects which are not present in the approximate calculations.

The coherent states CSI and CSII lead to a very good description of the ground-state energy of the system, in a variational sense, representing a remarkable improvement with respect to the Glauber coherent state. This was true even for a system with a small number of particles, in which case the mean-field approximation with coherent state (14) is generally not good. Also the time evolution, although missing important features such as decoherence, is quite well described by the same coherent states. It is clear that the deformed coherent states give a much better description of the time evolution of the system than the coherent state CS (14). On average, the deformed coherent states describe reasonably well the frequency in the oscillations of the instantaneous boson occupation numbers and the respective time averages. The agreement is remarkably better for a large number of bosons as may be seen by comparing figures 3(*a*) and 4(*a*) with 3(*b*) and 4(*b*). The performance of the Glauber coherent state is by far less good and was not explicitly discussed in order not to overload the paper with information lying outside the main issue.

As a next step it would be interesting to apply the deformed coherent states and techniques studied in the present paper, possibly modified according to the problem under study, to real physical systems. Another problem of interest is to study the inclusion of decoherence at the mean-field level.

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